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MEASUREMENTS

NEW TECHNIQUES AND APPLICATIONS

A chapter on measurements may seem inappropriate in a text claiming to be on theory. If we were to review current technology, which we won't, then it would be inappropriate. What we will do is utilize some of the new techniques that we have discussed in this text, things like T-matrix and modal radiation, to see if there might be new ways of looking at the measurement problem. Of course we will find that there are.

12.1 Philosophy of Measurement

Why does one make measurements? In the case of a transducer it is usually to ascertain the performance of the device. We hope that the knowledge learned from this exercise is sufficient to fully describe the performance of the device. In mathematical language, we seek to measure a set of functions which span (fully quantifies) the space being described. When dealing with nonlinear systems with time varying parameters whose final usage involves a subjective judgment we will probably never know if the measurements truly span the space. The end product of our designs being subjective perception poses a significant problem, one that we will not delve into any more than we already have. From a scientific point of view, based on what we have described in this text, we will define a minimum set of measures that are required to quantify those aspects of the problem that we have studied. Whether or not these measures are necessary and sufficient from the subjective view point is not know, or at least not to us. However, they will certainly a good starting point.

Those measurements that we propose being taken are:

- the frequency response of the linear system
- the polar response of the linear system
- the nonlinear transfer functions for all significant orders
- the frequency response as a function of thermal load
- the thermal time constants

Notably missing are time domain measurements like waterfalls, impulse response, etc. This is because of what we know about the Fourier Transform from Chap. 1, where we described how there is nothing that can be seen in the time domain of a linear system that cannot be discovered in the frequency domain.

Agreed, some things may be easier to see or do in one domain versus the other but nothing new can be seen. In our experience the frequency domain is the more straightforward to understand and to implement.

If we were to measure all of the above information, we would have a pretty good picture of how a transducer performs, at least in the configuration that we measured it. This simple fact – that any measurement is only valid in the configuration that it was made – is extremely limiting, for it does not allow us to utilize these measurements in a design context. This is why it is more desirable to use a parametric approach to the quantification problem. For example, consider lumped parameter parametric representation of the fundamental resonance of a loudspeaker – the Thiele-Small parameters or the mass, compliance, Bl, etc. – are useful in the design process. However, the frequency response is far less so. The reason for this is that we know, from theory, how the parametrized performance of the loudspeaker will change in a given circumstance because we know how the parameters affect and are affected by the system. With a frequency response curve, we do not know how it will change if this driver is put in a particular system configuration, like a ported box. It would be extremely useful to be able to parameterize all of our measurements so that we could have the same utility for all of them as we do in the example above. This is doable to certain extent and it is our goal in this chapter to show how.

12.2 Measuring the First Mode Parameters

Once again, we will find our T-matrix approach to be useful. Eq.(12.2.1) shows the T-matrix representation of a transducer. As before the individual matrices are progressively multiplied together, just as they appear in the device, to form a single matrix which represents the Device Under Test (DUT)

$$\begin{bmatrix} E(\omega) \\ I(\omega) \end{bmatrix} = \begin{bmatrix} 1 & R_e - i\omega \cdot L_e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & Bl \\ Bl & 0 \end{bmatrix} \begin{bmatrix} 1 & R_m - i\omega M_m - \dfrac{1}{i\omega C_m} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{S_d} & 0 \\ 0 & S_d \end{bmatrix} \begin{bmatrix} P(\omega) \\ U(\omega) \end{bmatrix} \tag{12.2.1}$$

Multiplying out the T-matrices results in

$$\begin{bmatrix} E(\omega) \\ I(\omega) \end{bmatrix} = \begin{bmatrix} \left(R_e + i\omega L_e\right)\dfrac{S_d}{Bl} & \dfrac{-L_e M_m \omega^2 + i\omega\left(L_e R_m + R_e M_m\right) + \left(R_e R_m + Bl^2 + \dfrac{L_e}{C_m}\right) + \dfrac{R_e}{i\omega C_m}}{Bl\, S_d} \\ \dfrac{S_d}{Bl} & \dfrac{\dfrac{1}{i\omega C_m} + R_m + i\omega M_m}{Bl\, S_d} \end{bmatrix} \begin{bmatrix} P(\omega) \\ U(\omega) \end{bmatrix} \tag{12.2.2}$$

Since the radiation impedance for any configuration is fixed it should be noted that the two output variables are not independent. The pressure and volume velocity are always related by the known (or determinable) radiation impedance Z_{load} as

$$Z_{load}(\omega) = \frac{P(\omega)}{U(\omega)} \tag{12.2.3}$$

which results in the ability to eliminate either the pressure or the volume velocity in Eq. (12.2.2).

The measurement techniques that we will develop in this section are independent of the form of the load placed on the DUT. Different load conditions result in slightly different equations. However, the analysis procedure remains virtually identical. Some of these different loads may be dictated by the DUT. For example a hearing aid transducer does not radiate sufficient sound to allow for a free space measurement, hence they are most effectively measured using a small tube or volume, much as they are used in practice. In this situation, the impedance for this load is simply substituted in the equations in a manner identical to that which we will be showing shortly. Further, some loading conditions may also give better results in particular situations. A noisy lab, or manufacturing plant may be better suited to the use of a closed box load where the microphone for measuring the radiated sound pressure can be placed inside the box. The availability of an anechoic chamber or the requirement to use a standard setup may dictate other loading conditions and test configurations. The techniques described here are applicable to all of them.

As a first example, we will use the configuration where the measured acoustic variable is the pressure inside of a closed rigid box, of known volume *V,* which contains the DUT. It can be shown that the pressure and volume velocity inside of the box will be related as

$$P(\omega) = i\omega \frac{\rho c^2}{V} U(\omega) \tag{12.2.4}$$

where we have assumed that the box is small enough so that modal considerations are not required. Of course, we could use the results of Chap. 5 to find a more exact relationship between $P(\omega)$ and $U(\omega)$ in a sphere or a cylinder, if this is deemed to be advantageous. At this point, it appears to us that the simple form shown above is satisfactory.

Using Eq. (12.2.4) in Eq. (12.2.2) to eliminate the diaphragm volume velocity yields the following two equations

$$E(\omega)=\left(R_e\frac{S_d}{Bl}-\frac{i\omega R_e M_m-\left(R_e R_m+Bl^2\right)+\dfrac{R_e}{i\omega C_m}}{Bl\,S_d}\right)\frac{-i\omega\rho c^2\,P(\omega)}{V}$$

$$I(\omega)=\left(\frac{S_d}{Bl}-\frac{\dfrac{1}{i\omega C_m}-R_m+i\omega M_m}{Bl\,S_d}\right)\frac{-i\omega\rho c^2\,P(\omega)}{V} \tag{12.2.5}$$

where we have also dropped the terms containing the inductance. Dropping the inductance from these equations is accurate if, and only if, the inductance is not a significant factor at resonance and slightly above. If, as can happen, the inductance is a significant contributor near resonance then it must be included in the analysis. While this is a complication, the extension to the development shown here is not difficult.

Rewriting Eq. (12.2.5) in terms of transfer functions from the voltage and current to the pressure in the box yields

$$\frac{P(\omega)}{E(\omega)}=\frac{\dfrac{Bl\,S_d C_{ab}}{C_b R_e}}{1-M_m C_{ab}\omega^2-i\omega\left(R_m+\dfrac{Bl^2}{R_e}\right)C_{ab}}=\frac{f}{1-e\,\omega^2-i\omega b}$$

$$\frac{P(\omega)}{I(\omega)}=\frac{\dfrac{Bl\,S_d C_{ab}}{C_b}}{1-M_m C_{ab}\omega^2-i\omega R_m C_{ab}}=\frac{g}{1-e\,\omega^2+i\omega d} \tag{12.2.6}$$

where

$$C_b=\frac{V}{\rho c^2} \qquad\qquad C_{ab}=\left(\frac{1}{C_m}+\frac{S_d{}^2}{C_b}\right)^{-1}$$

The last two equations are simply the acoustic compliance of the test box and the mechanical compliance of the DUT as installed in the test box. Eqs. (12.2.6) represent readily obtainable transfer functions from the input voltage and current to the pressure inside of the box. Even simpler equations result by substituting constants b, d, e, f and g, as shown in the right hand sides of Eqs. (12.2.6), for the parameters of the DUT (i.e. R_{ms}, R_e, Bl, …). The set of variables b, d, e, f and g uniquely define the parameters of the DUT (R_e, ... S_d).

By using a standard method of nonlinear curve fitting (nonlinear because the terms are not linear in ω not because the system in nonlinear) such as the Levenburg-Marquardt technique[1], the values of the constants b ... g can be fitted to the data from the measured transfer functions. The usual form of the Levenburg-Marquardt algorithm uses real data, but our data is complex. It is straightforward to modify common routines for complex data.

There are five equations in six unknowns. To define the parameters of the DUT from the fitted values of $b \ldots g,$ we need either another measurement or another equation. Fortunately, the radiating area, S_d, is usually known, or is certainly easily measured, although there is always some uncertainty as to what constitutes this area when a simple radius measurement is used. With a know value for S_d a sufficient number of equations exists to allow for the unique determination of the other five.

From the values of the constants, we get

$$Bl = \frac{(b-d)}{fC} \qquad M_m = \frac{e(b-d)}{fgC^2} \qquad R_e = \frac{g}{f}$$
$$C_{ab} = \frac{fgC^2}{(b-d)} \qquad R_m = \frac{d(b-d)}{fgC^2} \tag{12.2.7}$$

where

$$C = \frac{V}{S_d \cdot \rho \cdot c^2}.$$

The moving mass of the system, M_m, includes the radiation load and the mechanical compliance C_{ab} includes the compliance of the rear box placed on the DUT.

An alternative approach to measuring the pressure in a closed box of known volume is to measure the nearfield pressure of the transducer. The relationship between the pressure and volume velocity in this case is also known[2]

$$P(\omega) = \frac{i\omega\rho a}{S_d} U(\omega) \tag{12.2.8}$$

$$a = \textit{the radius of the diaphragm of the DUT if it is round}$$
$$= \sqrt{area / 2\pi} \textit{ if it is not round}$$

This change in the manner of measurement of the pressure changes the transfer functions into high pass functions (as opposed to the low pass functions found in the closed box measurement). Following through on the derivation, as before, yields for the parameters of interest

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1. See Press, *Numerical Recipes*, Chap. 14.
2. see Kinsler, *Fundamentals of Acoustics*, Eq. 7.63 with $r=0$ and small ka

$$Bl = \frac{(b-d)}{f}\rho a \qquad M_m = \frac{e(b-d)}{g\,f}(\rho a)^2 \qquad R_e = \frac{g}{f}$$
$$C_m = \frac{g\,f}{(b-d)(\rho a)^2} \qquad R_m = \frac{d(b-d)}{g\,f}(\rho a)^2 \tag{12.2.9}$$

In this configuration, the calculations require an accurate knowledge of the radius a whereas in the prior approach, we require an accurate knowledge of the volume *V*.

It is also possible to eliminate the pressure $P(\omega)$ by using the relationship between the far field pressure and the cone volume velocity. The algebra is straightforward and leads to a nearly identical set of equations as the nearfield case.

A plane wave tube can also be placed over the DUT and a microphone used to measure the pressure in the tube. In this case, the transfer functions take on a bandpass form, but the rest of the analysis is the same. Any technique that uses a pressure signal will yield equations which contain undetermined coefficients of a second order function. In practice, the differing styles of the fitting functions and the quantities that we must know to apply them may result in better resolution of the parameters for one configuration over another, for a particular type of device.

The radiating surface volume velocity can also be calculated by the use of a two microphone output measurement in a plane wave tube. The difference of the two microphone measurements is proportional to the gradient of the pressure, or the volume velocity. The advantage of the two microphone technique is that it calculates the volume velocity directly ignoring the standing waves in the tube.

Another interesting loading configuration is that of the use is that of a fixed plate clamped onto the front of the radiating transducer. In this configuration the load can be assumed to be that of a very large resistance and that the volume velocity of the transducer will be zero. This simplifies Eq. (12.2.3) to

$$I(\omega) = \frac{Bl}{S_d}P(\omega) \tag{12.2.10}$$

or

$$\frac{P(\omega)}{I(\omega)} = \frac{S_d}{Bl} \tag{12.2.11}$$

From this equation, it is clear that this complex spectral ratio *should* be a constant. If the transfer function does not produce a flat spectrum then the assumptions of zero volume velocity have probably been violated. This can occur for DUTs with significant leakage around the radiating member. If we actually want to know what this leakage resistance value is, then an analysis of the situation would show that the transfer function defined in Eq. (12.2.11) would be a simple first order HP filter which could be fit to find the parameters. If there is a range of approximately flat response then the value of this response is the ratio of the radiating area S_d and the drive constant Bl. In this way, we can indirectly measure the radiating area S_d.

We have seen that the use of the T-matrix approach leads to an entirely different procedure for calculating the parameters of a device's principal resonance. Unlike traditional perturbation methods (added mass or added compliance), which can have significant problems creating consistent results, this new technique does not perturb the system at all. All of the required data can be taken at one time with no user intervention required. The new techniques can be adapted to a number of different configurations and can even be used in situ without access to the driver itself.

At this point, we need to discuss (some more) why device parameters that we have just measured are useful and the limitations of their application. These parameters define the lowest mode of vibration of a transducer. All transducers have this mode. It is of fundamental importance to any transducer. However, these parameters can only be expected to provide valid results as long as the model on which they are based remains valid. In every case, this underlying model has limited scope. The model is correct only in the region where the system acts as a single degree of freedom mechanical vibrator. At some point, higher frequency modes enter the picture from either the mechanical, or acoustical systems and can cause a major disruption of the validity of the assumed single degree of freedom model. The device is usually said to be in the modal "break up" region. Only when the effects of these higher modes are out of band does one have a fairly complete model of the device with the simple parameters that we have been looking at.

None-the-less these parameters have been shown throughout this text to be of useful in analyzing and designing systems utilizing transducers. The question certainly arises as to how we could extend these models to higher frequencies, hopefully, in a parametric way, so that we might broaden their range of applicability. Traditionally, high frequency cone analysis has been done by modeling it, and sometimes its supporting mechanism, with FEA or some other numerical approach. The sound radiation is then calculated with a Boundary Element calculation or some other comparable technique. These approaches can give good results, but they rely on models with hundreds to thousands of degree's of freedom. While accurate this approach does not lend itself to parameterization.

We should also ask the question as to just how far into the modal "breakup" region we want or need to go. The fact is that once a transducer begins to break up, it tends to do so rather quickly and almost always with a pronounced effect on the radiated output. For many, if not most designs, once the break-up region has been reached the device is no longer useful. A notable exception to this rule is the automotive application, where there is seldom enough room to use multiple drivers. These applications are very specific and owing to the huge volumes of these products, the intensive analysis techniques described above are useful, but for typical hi-fi or home theatre systems we only need to quantify the transducers performance to just above the first few break-up modes. We will find in the techniques that we will develop that they are applicable well into the break-up region. However, the further we go into this region, the more complex we will find the analysis (more parameters are required for specification and the calculations become more in-depth). The principal advantage of the new techniques will be the fact that they are very effective for a small number of modes.

We will now appear to digress a bit and define the polar measurement techniques that we will use, but along the way we will find the parametric modeling approach that we are seeking. The two things will turn out to be one and the same.

12.3 Polar Response Measurement

We have not talked about the measurement of frequency response other than the parametric approach of the previous section, which we know is only good at lower frequencies. When someone discusses the frequency response of a transducer, they usually mean the axial response, a single spatial point. It will turn out that this single spatial point does have some particular significance, but for now, we will consider it to be just one of so many spatial points.

Seldom is a single transducer the desired end result we need to combine multiple devices together to create a system. We could measure the frequency response of the system at a multiplicity of spatial points (hopefully with some scheme in mind) and store this data as a representation of the system response. This approach is not that uncommon. On the other hand, we could use some of the results that we have found in this book to dramatically simplify the system specification by reducing the full system into a number of smaller subsystems. It is usually the case that the subsystems or individual transducers have some symmetry, even though the completed system may not. We can use this fact, and others, to reduce the complexity of specifying a loudspeaker system.

We know from Chap. 8 (on arrays) that sources can be combined by considering two vectors when summing their response. The first is a translation from some fixed point and the second is a rotation about some fixed vector. With these two vectors defined, we can combine the polar responses of individual subsystems (drivers) into a model of the complete system. If we can utilize any form of symmetry in the specification of the subsystem's polar response, then we can effect a substantial reduction in both measurements, as well as data.

The translation of a subsystem is accomplished through the use of the three space Green's Function (see Table 3.1, on page 66). The function that is used in this case is simply the vector difference between the Green's Function from the new source location to the field point and the Green's Function from the origin to the field point. We have already seen applications of this technique so we won't elaborate on it here. Multiplying every data point in the polar response by this complex number will yield the correct amplitude and phase for the sound radiation from a source that is no longer located at the origin where the data was taken.

Rotation is accomplished by a simple reorganization of the reference angle in the polar data set. The polar response for a system of sources is then the complex sum of the pressures from the individual subsystems.

The above procedure will yield an accurate representation of a loudspeaker system given each subsystems polar radiation pattern (polar response, including axial) – a vector to its location relative to some common fixed origin of the system and its orientation angle relative to some common fixed vector orientation. If there is no symmetry in the subsystems to exploit then the resulting data set is actually larger than the data set for the original system. Fortunately, virtually all transducers have some form of symmetry, either axial or quadrant etc. Additionally, we have not yet utilized our knowledge of modal radiation for this problem.

There is one area where the above approach is not accurate, and that is that it neglects diffraction effects. Diffraction is not an easy problem to deal with, but it is doable. Diffraction is never a good thing and should always be minimized in a good design. Therefore, if we are dealing with systems for which diffraction effects of box edges etc. have been minimized, then our approach is quite viable and accurate. If, on the other hand, we want a model that includes these diffraction effects then there simply is no way to reduce the problem down to a more simplified parametric solution. From the recent, well published successes of *stealth* technology for airplanes, we know that diffraction is a problem for which there are solutions. For this reason, we opt to take the path which requires us to ignore the diffraction and to minimize it in our designs in order to make these models more accurate.

We know from Chap. 4 that we can represent sound fields and polar responses by radiation modes. We also know that there are a number of different geometries in which we can do that. We studied two planar geometries rectangular and circular and two three dimensional geometries – spherical and cylindrical. There is also an elliptical capability which could be developed that would involve kernels of Mathieu Functions. Owing to the limited usage of elliptical sources we did not develop this geometry. In the elliptical case, either the rectangular or the circular source could also be used, but with less efficiency. We would have to extend the circular source equations to allow for non-axi-symmetry (see Sec. 4.5 on page 83). In this case, we would expect the velocity to go to zero at the sides of the circle or the corners of the rectangle. In the end, the choice of geometry depends on the particular situation being analyzed. For most situations, the most efficient is to use one of the two planar sources, since nearly any source can be made up from these two geometries. The further the actual source geometry is from the chosen geometry, the slower the convergence of the results; more modes are required. We will develop the circular aperture in depth and discuss the rectangular aperture briefly. We will discuss the spherical and cylindrical geometries. However, the analysis that we will go through is the same for any geometry.

Recall that we can expand the radiation response from a circular aperture into an equivalent set of functions of the form

$$P(\rho) = 2\pi a \sum_m A_m \frac{\rho a\, J_1(\rho a)}{(\rho a)^2 - (\pi \alpha_m)^2} \tag{12.3.12}$$

$\rho = ka\sin(\theta)$

θ = *angle to the field point at which the pressure is measured*

where the other terms are all know from Chap. 4. This equation is equivalent to a Hankel Transform. Of particular interest is the lowest order term, $m = 0$. In this case, we have

$$P(\rho) = 2\pi a \sum_m A_m \frac{J_1(\rho a)}{\rho a} \tag{12.3.13}$$

From Fig. 4-4 on page 76, we know that this is the only term with a value at ρ=0, (on axis). This is an important fact for it indicates the significance of the axial response. The axial response is the value $A_0(\omega)$ in Eq. (12.3.12), which we have now written to specifically indicate that it is frequency dependent. Since the zero order mode is a piston mode or stated more mathematically, the average response across the aperture, then the axial response is the average response of the aperture across frequency. This would certainly make it the single most important point to quantify. All of the other terms simply account for polar variations in the response or deviations from the average velocity in the aperture. They are completely equivalent representations. What this means is that the velocity response in the aperture and the polar response in the far field are completely equivalent. Of course, we already knew this from Chap. 4, but now it takes on an even greater significance. If we can determine the velocity distribution in the aperture then we have all the data that we need for a complete specification of the polar response. Since we also know that it is possible to represent any velocity distribution in the aperture by its modes we can parameterize the polar response by parameterizing the apertures modal velocity response. We are one step closer to our goal.

We know from Fig. 4-4, that for a given frequency range, there are a fixed number of modes that contribute to the response and that each mode contributes primarily in a particular band of values of ρ. If we have the measured polar response then there should be a way to decompose the polar response into the aperture velocity modes. Theoretically, no more data points are <u>required</u> than the number of modes to be extracted, but additional data is always useful in the numerical calculations.

Consider a typical example of an axi-symmetric source. By taking polar response measurements along an arc from the axis to a mounting baffle, we can fit the coefficients A_m to the data at each frequency point. In practice this is not so easy since the equations tend to be singular at lower frequencies and can be near singular at other frequencies. Dealing with this problem is beyond the scope of this text other than to mention that this problem has a well known solution know as Singular Value Decomposition (which we also talked about in Sec. 1.14 on page 23).[3]

We should also note one other thing of importance. The modes not only have a sort of cut-in value of ρ but they also have a cut-out value. For instance mode one cuts in at $\rho = 3.83$ and cuts out at a value of $\rho = 10.83$ outside of this range This mode contributes only a small amount and there is little error in assuming that it is zero outside of these limits. It is exactly this characteristic that makes the calculations singular. Outside of a mode's contribution region, large values cause virtually no change in the result – almost the definition of a singular matrix. This characteristic also means that one need not consider the contributions of the mode outside of this region – we can band limit each higher order mode. Only the zero order mode will have full bandwidth. In the data calculation stage, it is wise to exclude a mode from the calculation when it is outside of its contribution bandwidth because the calculations will be less singular and therefore more stable.

An alternate point of view of what we are proposing here is interesting. The modal model gives us, in essence, a set of vibration modes of some virtual planar circular source that yields exactly the same polar radiation field of the actual device. These vibration modes are, if not the same as, very similar to, the DUT vibration modes that occurred in order to produce the radiated sound field that was measured. So, in effect, what is being developed is an equivalent model of the radiated sound field as a source vibration pattern – a modal velocity analysis performed with radiation data. The knowledge that we can readily recreate the radiated field from this data set by using simple radiation formulas applicable to these modes makes this attractive. Now comes the best part – since the source has much smaller physical dimensions than the radiated sound field itself, the data required to accurately describe it is substantially less than that required to describe the radiated field. In performing this data reduction, we have found an ideal way to describe a polar radiation map in the simplest possible form. We know exactly what the maximum order in the modal summation is required to obtain the frequency bandwidth that we desire and exactly how much data that we need to describe these modes. Being parametric, we have infinite resolution with an accuracy which we can control at will.

If the $A_m(\omega)$ were independent of frequency, then the result of the above procedure would simply be a single complex number for the amplitude of each radiation mode – above the fundamental vibration mode which have already quantified. There would only be five numbers from which we could reconstruct the entire polar radiation pattern. In the general case, however, resonance modes of the acoustic cavity in front of the transducer, as well as mechanical vibrations in the structure itself, will cause the radiation mode coefficients to be frequency dependent.

When the $A_m(\omega)$'s are frequency dependent then it is desirable to accurately model this frequency dependence with as few coefficients as possible to keep our

3. See Press, et.al. *Numerical Recipes*, Chap. 3

model as small and manageable as possible. There are well known techniques for doing this frequency modeling – for example, using Auto Regressive-Moving Average (ARMA) models, as we introduced in Chap. 1[4]. We could also use an FFT response, although it is well known that the FFT is not an efficient usage of data. We could use different techniques for different modes. This would allow, for example, the axial response, which depends only on the lowest order mode, to be represented with say, an FFT for high resolution, while the higher order modes that contain the variations of this average response with angle could then be represented with a simpler frequency model, like ARMA. All of these frequency-modeling techniques are well known and not critical to our discussion here. The important point is that when combined with the radiation modal expansion techniques, these data reduction techniques represent an extremely efficient method for reducing the total amount of data required to accurately describe the polar response of a radiating system. Data reductions of several orders of magnitude are possible.

There is a good psychoacoustic reason to model the frequency response functions with parametric models. The reason for this is because we know, from the work of Toole, that the ear is most sensitive to loudspeaker aberrations whose "area" is greatest. By area we mean the base times height of the aberration relative to the baseline. This is not a precise thing to define, but it is not necessary to do so. The parametric spectral estimation techniques for ARMA models (see Marple and Chap. 1) react in exactly this same manner. They use their degrees of freedom to "go after" those features that are most predominant ones first, i.e. those with the largest "area", until the degree's of freedom have been exhausted. The details that the model leaves behind are exactly those details that the ear doesn't hear. One might even be able to find the order of the model (the values of P and Q in Eq. (1.14.49)) that best represents the ears acuity, and use them in the fitting algorithms.

If the frequency modes are extracted parametrically as pole-zero models then we have achieved exactly what we were seeking – a parameterization of the high frequency response which blends in well with the existing parametric models of the fundamental resonance.

When modeling the frequency response of the radiation modes it is unnecessary to model the low frequency high pass filter response that is a characteristic of every transducer. This response is well characterized by the standard parametric models and need not be duplicated. The response of interest here is the deviation of the actual response from the passband efficiency predicted by the standard models. Thus, we would remove the low frequency effects from the axial response and model only the upper frequencies normalized to the passband level from the parametric model. Then we would normalize the off axis response to the axial response, i.e. remove the average frequency response from the polar data. Finally, this means that the high frequency models for the polar modes will

4. See Marple, *Digital Spectrum Analysis With Applications*

all be flat (0 dB) at lower frequencies. Low frequency modes (long impulse responses) are notoriously hard to model with typical time domain data reduction techniques (like ARMA, etc.). Lastly, the higher order modes need only be modeled within each of their individual contribution bands.

Let's summarize the procedure:

- Take data at points along an arc from the axis to the baffle in at least the number of point as the number of modes in the expected expansion. More points is useful.
- Model the axial data to whatever order is required.
- Rewrite the data in the variable $ka \sin \theta$.
- Normalize all the remaining data by the axial data.
- Expand the remaining data in the polar modes either simultaneously or one at a time (each technique has its advantages and disadvantages. Only use data that is pertinent to the mode under consideration.

The data obtained can be used as part of a system model or as a standalone model for use in computer simulations etc. The data is so condensed that it could be stored in any number of convenient formats including bar codes, etc. Unlike a full blown polar measurement (which is an enormous amount of data), this model could be efficiently transmitted over the net.

We will briefly touch on the use of a rectangular geometry of radiators, such as we would have for a square mouth horn or a square array of sources. Because there is no axi-symmetry, we need to take data in a slightly different geometry. Instead of the polar angles of ϕ around the disk and θ off the axis we need data on two orthogonal circular arcs (see Fig. 4-8 on page 78), which intersect at the axial normal vector from the sources center. It is best if these two arcs are parallel to the edges of the assumed rectangular source. In this application, the polar data is utilized in two variables, $z = ka \sin(\theta_x)$ for the arc parallel to the edge of length $2a$ and $z = ka \sin(\theta_y)$ for the arc parallel to the edge of length $2b$. The transformed polar data is then fit to the m terms using

$$g_m(z) = \frac{z \sin(z)}{(m\pi)^2 - z^2} \qquad (12.3.14)$$

for each arc separately.

In general, rectangular sources will have a two dimensional array of coefficients as shown below

$$\begin{matrix} A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} \\ A_{1,0} & A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,0} & A_{1,2} & A_{2,2} & A_{2,3} \\ A_{3,0} & A_{1,3} & A_{3,2} & A_{3,3} \end{matrix}$$

(where only 16 terms have been shown). The measurements described above will only determine the coefficients with a zero (0) in them, namely, the first row and first column. To obtain the other coefficients we need data at angles (in the x-y plane) between those already taken. As a simplification consider the angle $\varphi = \tan^{-1}(b/a)$. By taking data along an arc in a plane at this angle relative to the x axis and then transforming the data by using

$$z = k\left(\frac{a \cdot b}{a^2 + b^2}\right)\sin(\theta) \tag{12.3.15}$$

θ now being the angle away from the z axis along this new line in the x-y plane. The diagonal coefficients in the above equation can then be determined using Eq. (12.3.14) to fit the data not already accounted for by the zero terms that we have already calculated. In general, in order to determine the coefficient $A_{2,1}$ we would take data along a line at an angle of $\varphi = \tan^{-1}(2b/a)$. The general procedure is easy to see from these examples.

In the general case the expansion is done as

$$G_{m,n}(X_m, Y_n) = g_m(X)g_n(Y) \tag{12.3.16}$$

$$X = ka\sin(\theta)\cos(\phi) \textit{ for the } \mathrm{x} \textit{ direction}$$

$$Y = kb\sin(\theta)\sin(\phi) \textit{ for the } \mathrm{y} \textit{ direction}$$

The functions $g_m(z)$ are given in Eq. (12.3.14). As in all other cases, modeling the frequency dependence of the radiation mode coefficients by means described above can reduce the data load.

A source that is best represented as a section of a cylinder, for example, a line array or a source being used in a line array would use the cylindrical expansions. In this case, the radiation pattern would be expanded in terms of a series of sines and cosines, i.e. a Fourier Series for the angular direction, θ. If the source is symmetric around the cylinder then only cosine terms will exist. The vertical radiation is expanded in a set of functions $g_m(z)$ as shown in Eq. (12.3.14), except that $z = ka\sin(\phi)$. In this case, where a is the height of the source and ϕ is the angle away from the normal in the plane of the axis and the normal exactly like the rectangular case in one dimension. In general, there is a full matrix of coefficients and data needs to be taken at those points that yield the required information. The procedure is a direct extension of what we have already discussed above and its implementation is straightforward.

In essence, any finite source can be expanded in this manner but for certain common geometries, the previous examples are preferred, because, in a mathematical sense, they will converge more rapidly, resulting in a smaller number of radiation modes for equivalent accuracy. The radiation modal expansion for the spherical case is well described in the literature[5] and will not be elaborated on here.

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5. See Weinreich, *JASA*

We have now described efficient techniques for the specification of the first two measurements in our list. We will now move on to measuring the nonlinear response.

12.4 Measuring Component Nonlinearity

To start this section, we would like to point out that there are just about as many ways to measure the nonlinear parameters as there are to measure the linear ones. Each one has its pros and cons and in the end, if they are all accurate, then they should all get the same answer. In that case, it does not make much difference which one is used. We are simply discussing the one that we will show because it is not documented anywhere else. We neither claim that it is better than or as good as any of the other techniques. For good discussions of other techniques see the works of Klippel[6] and Park[7].

We want to discuss a means for measuring the nonlinear transfer functions for each of the significant components of a transducer, in a manner consistent with the analysis that we did in Chap. 10. In that chapter, we found that distortion could be represented by frequency responses of the various orders of the nonlinear transfer functions. We determined that this was an efficient way to look at the nonlinear problem since it eliminated the question about the types of signals to be used in studying the nonlinear response of the transducer system. We noted there and we will reiterate here, that our mathematical analysis is a simplified approach to the question of analyzing the transducer's nonlinear response, a sort of first order approximation. The exact nonlinear problem is very complex and in the case of a transducer, there can be far more nonlinear sources than the two that we looked at in Chap. 10. The fact does remain, however, that a simple technique which can be used to quantify these two (or perhaps three) principal nonlinear components using techniques and equipment that is readily available would be very useful.

We are hoping that our techniques produce results which, if not highly accurate, are at least a reasonable assessment of the nonlinear transfer functions for various components of a transducer. We will show, based on the results of Chap. 10, how to do this for the two components that we have analyzed, stating once again, that the techniques can be readily extended to other components.

There are a number of ways to get the transfer function of the various orders of nonlinearity. Almost any technique that is used to measure the linear transfer functions will work for the nonlinear ones. The theory of nonlinear systems, in its strictest sense, only applies to Gaussian noise inputs. This is because the statistics of the expected value, which is central to all systems analysis, can only be applied to random inputs with normal distributions (at least not without a lot more work). For our purposes, it is not critical that we exactly measure the nonlinear frequency

6. Klippel, Various *JAES* articles
7. Park, *JAES*

responses since, after all, we are only getting an estimate of the component nonlinearity.

The only thing that we have to be careful of in making our measurements is scaling the inputs to what we will call x_{peak}. The value of x_{peak} is most easily estimated for sine waves since we can easily calculate the excursion from a given SPL and cone area. We must also keep in mind that for us to apply the analysis of Chap. 10, we need to know the values of the a_n's, which are not the amplitudes of the harmonics but the amplitudes of the higher order Legendre Polynomial displacements. Recall that

$$\begin{aligned} a_0(\omega) &= x_0(\omega) \\ a_1(\omega) &= x_1(\omega) \\ a_2(\omega) &= \tfrac{3}{4} x_2(\omega) + \tfrac{1}{4} x_0(\omega) \\ a_3(\omega) &= \tfrac{5}{8} x_3(\omega) + \tfrac{3}{4} x_1(\omega) \end{aligned} \tag{12.4.17}$$

With pressure measurements only, it is impossible to get the values of $x_0(\omega)$ since they do not radiate sound. We will only be able to determine the second order nonlinearity up to a constant displacement of the origin of these curves. This is not a serious problem and can be eliminated with a measurement system based on displacement such as a laser. We will talk primarily about assessment of the nonlinear curves through pressure measurements, since this is so convenient.

We can actually get the linear response, the first mode parameters and the mode parameters for the higher modes of the first radiation mode, the nonlinear component parameters and the thermal variations of the linear parameters all with a single setup and a continuously sampled set of data which can be broken down into numerous subsets or records of the data for analysis. The length of the total data, the number of records and the record length will all depend on the accuracy and frequency resolution desired. These are fairly standard items and will not be elaborated on here.[8]

We will discuss the most straightforward method and leave the variations as an exercise. The easiest method to describe is to consider the block diagram shown below where

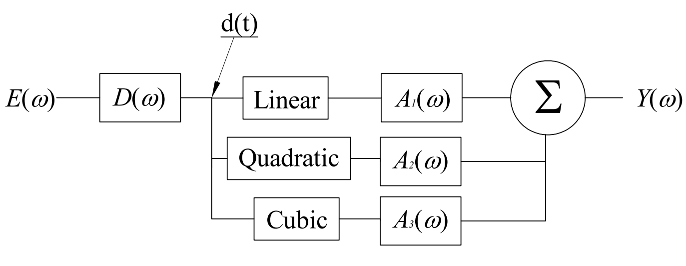

Figure 12-1 - Block diagram for a nonlinear system

8. See Bendat and Piersol, *Engineering Applications of Correlation …*

$E(\omega) = input\ signal\ spectrum\ (voltage,\ measurable)$
$d(t) = displacement\ signal\ (not\ measurable)$
$Y(\omega) = output\ signal\ Spectrum\ (pressure,\ measurable)$

(Please excuse us for mixing our domains in this figure. It simply makes the discussion easier.) The blocks for the orders are assumed to be zero-memory in this application. The frequency dependence is contained in the transfer function $A_n(\omega)$.

If we can determine the transfer functions $A_n(\omega)$, then we can plot out the nonlinear characteristics for the transducer at any frequency. The transfer function from the voltage to the diaphragm displacement, $D(\omega)$, has been assumed to be the same for all of the orders. From Chap. 10, we know that this is not the case. However this simplification does not generate a great deal of error since these functions are nearly identical, as Fig. 12-2 shows.

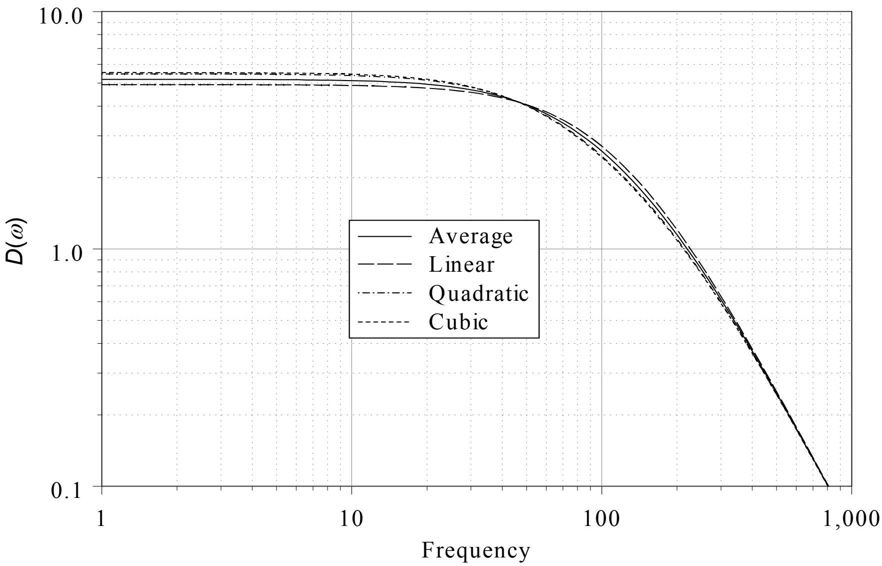

Figure 12-2 - Transfer functions for orders 1,2 and 3 and the average function

We can make these curves identical by using an average denominator for all of the curves. The error in this later case is no greater than about 5% for any of the functions. A common $D(\omega)$ makes the analysis much simpler.

If we now look at the system in Fig. 12-1 from $d(t)$ to the right we can see that we have a classic multiple input/single output (MI/SO) system with an observable $d(t)$ (actually it is only calculable as $F^{-1}\{E(\omega)\bullet D(\omega)\}$). It can be shown[9] that the system of Fig. 12-1 can be transformed to

9. Bendat, *Nonlinear System Analysis & Identification*

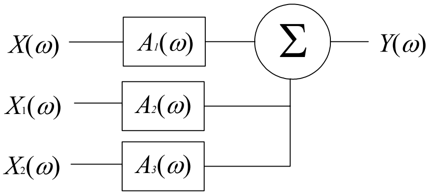

Figure 12-3 - Modified MI/SO for nonlinear system

$X(\omega)=E(\omega)D(\omega)=$ *the linear spectrum of the diaphragm displacement*

$X_1(\omega)=F(d(t)^2)$ *with the linear effects of* $X(\omega)$ *removed*

$X_2(\omega)=F(d(t)^3)$ *with the linear* $X(\omega)$ *and quadratic* $X_1(\omega)$ *effects removed.*

$Z(\omega)=Y(\omega)/\omega^2$ *the measured pressure signal transformed into a diaphragm displacement.*

The signals $X(\omega)$, $X_1(\omega)$ and $X_2(\omega)$ are now all uncorrelated and the systems analysis is straightforward. This de-correlation process is well known[10] and these functions are readily calculable from a series of time records of the input and output. The calculations are, of course, complex and best left to a computer. With the transformations as shown the function $A_1(\omega)=1$.

Once we have determined the $A_2(\omega)$ and $A_3(\omega)$ we can use these functions as follows. First find a parametric form for these functions as

$$A_2(\omega) = -i\omega c_{21} + c_{22}$$
$$A_3(\omega) = -\omega^2 c_{10} - i\omega c_{11} + c_{12} \tag{12.4.18}$$

where

$$c_{21} = \left(\tfrac{4}{3} Bl_0 + \tfrac{8}{7} b_2\right)\frac{b_1}{R_e}$$

$$c_{22} = \tfrac{2}{3}\left(\frac{b_1}{R_e} - k_1\right)$$

$$c_{10} = \tfrac{-14}{35}\left(\tfrac{16}{7} b_2^2 + \tfrac{128}{21} Bl_0 b_2 + 4Bl_0^2\right)\frac{b_1^2}{R_e}$$

$$c_{11} = \left(\tfrac{4}{5}\frac{Bl_0}{R_e} + \tfrac{8}{15}\frac{b_2}{R_e} + \tfrac{2}{5}\right)\frac{b_1^2}{R_e} - \left(\tfrac{8}{10} Bl_0 + \tfrac{1}{40} b_2\right)\frac{k_1 b_1}{R_e} + \tag{12.4.19}$$
$$\cdots \quad \tfrac{4}{5}\left(Bl_0 + \tfrac{3}{5} b_2\right)\frac{b_2}{R_e}$$

$$c_{12} = \tfrac{-2}{525}\left(8k_1\frac{b_1}{R_e} - 8k_1^2 + 105k_2\right)$$

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10. Bendat and Piersol, *Engineering Applications of Correlation and...*

where the b's and k's are defined in Eq. (10.2.9) and Eq. (10.2.10) and Bl_0 and R_e are both determined from the linear first mode parameters extracted earlier.

This is an over-determined set of equation which can be solved in a number of ways. We will not go into this detail.

We have seen that we can use standard techniques in linear systems signal processing to extract the nonlinear dependency of the Bl product and the stiffness. The process is summarized below

- Take a continuous time record of data, the voltage input and the radiated pressure (nearfield or far) at a input level such that the peak excursion of the diaphragm is at the displacement x_{peak}.
- Calculate the function $D(\omega)$ at this level from the linear transfer function
- Transform the input data signal into a displacement signal $d(t) = F^{-1}\{D(\omega)\bullet E(\omega)\}$
- Create the signals $d(t)^2$ and $d(t)^3$
- Using MI/SO techniques determine the functions $A_2(\omega)$ and $A_3(\omega)$.
- Fit $A_2(\omega)$ to equation $-i\omega c_{21} + c_{22}$
- Fit $A_3(\omega)$ to equation $-\omega^2 c_{10} - i\omega c_{11} + c_{12}$
- Find the values b_1, b_2, k_1, and k_2 from Eq. (12.4.19)

There is nothing in this procedure that is not a standard technique in spectral analysis. The above process could be performed on a PC with a sound card and a microphone. Some post-data capture calculations are required but those are straightforward.

12.5 Measuring the Thermal Parameters

We will close this chapter by discussing how we can measure the temperature variations of a transducers parameters, as well as the thermal time constants of these variations. The procedure is actually quite simple.

Since we know the value of R_e at room temperature and we also know its change in value with temperature, by simply monitoring this value we will know, at any point in time what the voice coil temperature is. By continuously running a signal, usually noise, through the DUT, at a level which does not cause significant nonlinearity, and taking data, either continuously or periodically, we can calculate the parameters at various temperatures. This should be done over a sufficient time period so that all of the components heat up, preferably reaching a steady state temperature. The signal level and the final temperature should not be so high as to cause the DUT's failure. A few iterations of this techniques may be required to find the right level to make this happen. From this data, we can easily plot out the change in the linear parameters with temperature.

To find the thermal time constants we would slightly modulate the noise signal, preferably randomly, but at a very slow rate, such that there are no level modulation frequencies in the passband of the DUT's acoustic response. This would typically be below about 10 Hz for the level modulations. From the extracted data for the thermal dependence we can find, by cross-correlating these parameter changes with the level data, a thermal cross-correlation function, from which we can readily extract the thermal time constants.

These measurements are all quite straightforward.

12.6 Summary

We described a minimum set of measurements that one should take to quantify a transducer and we have shown how make these measurements. We focused on ways to parameterize these aspects of a transducers performance. This parameterization makes the analysis and storage of the data a far simpler task than current procedures allow. While these techniques are not currently common or readily available, their simplicity and usage of common signal processing techniques makes there implementation fairly straightforward.